ADDENDUM: COMPLETE PROOF FOR CASE II

For case II, the following fact about $\mathcal{S}_{-T}$ will be needed:

<u>Lemma:</u> *If policy y is excessive insurance and policy x is not excessive* insurance and both x and y are in $\mathcal{S}_{-T}$ then so is any full insurance policy z where for some type j

$$V_j(z) = \min (V_j(x), V_j(y))$$

Moreover, z is in the interior of $\mathcal{S}_{-T}$ if either of the following conditions also holds:

a) −T includes no customer of type j.

b) x is incomplete insurance.

<u>Proof of Lemma:</u> It is sufficient to prove the equivalent results for each $\mathcal{S}_{fc}$. Without loss of generality, suppose $V_j(x) = V_j(y) = U_j$ (since the set is comprehensive below, we can find a point of excessive insurance and a point of incomplete insurance on the lower of the two indifference curves). Suppose c is of type i and let $\mathcal{S}_{fc}$ be defined by the indifference curve U_c and iso-profit line π_c. Suppose z is not in $\mathcal{S}_{fc}$, then at least one of the points x, y must lie above indifference curve U_c; otherwise it would violate single crossing for z to be above. Moreover, both cannot lie above the indifference curve, for if they did then they would both have to lie on or below π_c and z would as well. Without loss of generality, suppose x lies

above c's indifference curve and y lies below it. (See figure A3). Since z is full insurance it is the least cost way for c to attain utility $U_i(z)$. Since z is above π_c and x is below π_c it must be that the utility $U_i(x) < U_i(z)$ (and thus i < j). But along the indifference curve U_j, c's utility must be monotonic; otherwise single crossing would be violated. Thus increasing insurance beyond full insurance only increases c's utility along the indifference curve U_j. This implies that the indifference curve U_j always lies above the indifference curve U_c in the region of excessive insurance, and so it cannot intersect policy y, a contradiction.

We have proved that z is in the set $\mathcal{S}_{fc}$. If a full insurance point is on the boundary of the set and j(c) > j then all points of incomplete insurance are outside $\mathcal{S}_{fc}$. (see figure A4). If j(c) < j then all points of excessive insurance are outside $\mathcal{S}_{fc}$. Therefore if z is on the boundary either j(c) = j or z = x.

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Proof for Case II: We now construct a self *enforcing deviation undermining* (C,s) when the cross subsidized contract is excessive insurance. (By the observation on IIC allocations, there will be only one cross-subsidized contract; it will be the contract with the most excessive insurance, and the worst types will be subsidized).

The parties to the deviation will be denoted T. T consists of one firm f plus two sets of customers in C which we will denote T_1 and T_2. T_1 consists of all customers c in C satisfying the following two conditions:

(i) The customer (weakly) prefers a policy with excessive insurance among the policies offered in s.

(ii) This policy does not subsidize the customer.

T_2 consists of all customers c in C satisfying the following two conditions:

(i) The customer (weakly) prefers a policy with full insurance among the policies offered in s. (Thus, if no full insurance policy is offered, T_2 is empty.)

(ii) For every $c^* \notin C$ such that the full insurance policy is on the boundary of $\mathcal{S}_{c^*}$, $j(c^*) > j(c)$. (see figure A4).

We let $\mathcal{J}$ denote the set of types of customers in $T_1 \cup T_2$. To construct the deviation which undermines (C,s) we begin by constructing an allocation for types in C which uses only policies which are invulnerable to infiltration and which satisfies a self selection constraint. We will call such an allocation a *feasible deviating allocation:*

The allocation $(z_j)_{j \in \mathcal{J}}$ is a *feasible deviating allocation* if it satisfies the following two constraints:

$$V_i(z_i) \geq V_i(z_j) \text{ for all types } i,j \in \mathcal{J} \quad \text{(self-selection)}$$

$$z_j \in \mathcal{S}_{-T} \quad \text{for all } j \in \mathcal{J} \qquad \text{(invulnerability)}$$

To construct an agreement which undermines (C,s) we must ensure that all deviating players receive a utility strictly greater than that achieved under strategy vector s. We first show that there is a feasible deviating allocation, which we will call the "baseline" allocation, which gives every member of the deviation greater utility and in which no deviating customer is subsidized. If this allocation is pareto optimal among feasible deviating allocations we will show that it generates the self enforcing deviation undermining (C,s). If not we will find a pareto optimal allocation which pareto dominates the "baseline" allocation (and therefore strictly pareto dominates the allocation under (C,s)) which generates the undermining deviation.

Lemma: There exists a feasible deviating allocation from (C,s) which satisfies the following conditions: 1) it is strictly preferred by all members of T, 2) no customer in the deviation is subsidized, 3) every policy in the deviation is full or excessive insurance.

Proof: First consider customers in set T_2. The simplest case occurs when their full insurance policy is in the interior of $\mathcal{S}_{-T}$. For then all customers in T_2 can be moved to a larger full insurance policy still in $\mathcal{S}_{-T}$. By choosing a policy sufficiently close to the initial policy, we can make the cost to the insurance company of this deviation arbitrarily small. By choosing a policy sufficiently close we can also insure that the policy is not preferred by any player not in T_2. (Recall that T_2 includes all players indifferent between their initial policy and the full insurance policy. Thus moving such individuals from inefficient policies to a full insurance policy can only improve profits for the firm.)

On the other hand, if the initial full insurance policy is on the boundary of $\mathcal{S}_{-T}$, then by condition (ii) of the definition of T_2 the boundary has steeper (negative) slope than the indifference curves of the members of T_2. Thus their utility can be increased within $\mathcal{S}_{-C}$ by moving slightly into the region of excessive insurance (Figure A4). By the same argument as in the previous paragraph, we can make the losses to the firm arbitrarily small in the process.

Next consider the least excessive insurance policy chosen by the best remaining type in set T_1. Call this player c. We know that c does not covet the new policy given to players of better type (had he done so, he would have been given that policy and improved the firm's profits in the process.) Moreover, since players of better type have had their policies improved, and the initial strategy was IIC, they do not covet player c's policy. Nor do we need to worry about the incentive compatibility problems for players of lower type: Their policies are more costly to the firm, anyway, therefore it will only improve the firm's profits to have them move. Thus we can improve the utility for player c without *violating self selection*, as long as there are policies preferred by player c which lie in $\mathcal{S}_{-T}$. This is certainly the case as long as the initial policy is not on the boundary of $\mathcal{S}_{-T}$; it is also the case as long as the only customers c' in −T such that the initial policy is on the boundary of $\mathcal{S}_{fc'}$, are customers of worse type than c, for then preferred policies which reduce insurance will be in $\mathcal{S}_{-T}$. (See figure A5).

==================== INSERT FIGURE A5 ====================

Therefore we will show that the initial policy cannot be on the boundary $\mathcal{S}_{fb}$ for b of equal or better type. There are two possibilities: first, that b is in C but not in T. Since the deviation (C,s) was IIC, the c's policy can only be on the boundary if b is indifferent between it and his own. But in that case b would be part of the set T_1. The other possibility is that b is not in C. Then b must be of strictly better type: if b and c were of the same type then there would be a player in (C,s) whose outcome had not strictly improved over ONSA. In ONSA b's policy makes zero profits. For c to be on the boundary of $\mathcal{S}_{fb}$ c's policy must make 0 or negative profits if taken by b. Therefore it must make strictly negative profits when taken by a worse type, contradicting part (i) of the definition of T_1. We conclude c's policy cannot be on the boundary of $\mathcal{S}_{fb}$ for any non member of T of equal or better type. Therefore it is possible to improve c's utility without hurting the firm's profits, and we move him and all other players of his type to such a policy.

We then repeat this argument for each successive lower type, stopping only when we reach the first subsidized customer. Since dropping subsidized customers increases the firm's profits, the initial loss on customers in T_2 can always be made small enough that the net effect on firm profits is positive. This completes the proof of the lemma.

Let (b_j) denote the baseline allocation and let U_j denote the utility received by type j in the baseline allocation. Note that U_j is a non-increasing function of j, since all policies are full or excessive insurance. Next we look for feasible deviating allocations which maximize the firm's profit subject to the *reservation utility constraint:*

$$V_j(z_j) \geq U_j \qquad \text{for all } j \text{ in } \mathcal{J}.$$

Let π^* be maximum profits the firm can obtain among feasible deviating allocations satisfying the reservation utility constraints. The set is non empty, since the baseline allocation is in it and it is clear that a maximum exists.

<u>Lemma</u>: A necessary condition for the firm to attain π^* in a feasible deviating allocation satisfying individual rationality is for no deviating player to receive less than full insurance.

<u>Proof:</u> Suppose not and let j be the type in T who receives the most nearly complete of the incomplete insurance policies. Then profits will be increased by pooling j and all higher types in $\mathcal{J}$ at the full insurance policy x satisfying

$$V_j(x) = \max\,(U_j,\ V_j(z_k))$$

where k is the next lower type in $\mathcal{J}$. Since the baseline allocation satisfies IIC, pooling also satisfies IIC. Pooling also increases profits because this full insurance policy is less expensive than any incomplete insurance policy in the baseline allocation. Pooling does not violate the reservation utility requirements, since a full insurance policy gives the same expected utility to all types receiving it, and the reservation utility of j is greater than the reservation utility of any greater type in $\mathcal{J}$. Finally by the lemma on the shape of the set $\mathcal{S}_{-T}$ this full insurance policy is also in $\mathcal{S}_{-T}$ and therefore invulnerable to infiltration.

<u>Lemma:</u> There exists a pareto optimum among feasible individually rational deviating allocations in which the firm receives π^* and there is no cross subsidization.

<u>Proof:</u> Let $\mathcal{J}$ be the set of risk types in T. Consider the following maximization problem:

Problem R: Find $(z_i)_{i \in \mathcal{J}}$ to maximize

$$\min_{i \in \mathcal{J}} \{V_i(z_i)\}$$

subject to

$$\Sigma \, \mu_i \pi_i(z_i) = \pi^*$$

$$V_i(z_i) \geq V_i(z_j) \text{ for all } i,j \in \mathcal{J}$$

$$z_j \in \mathcal{S}_{-T} \text{ for all } j \in \mathcal{J}$$

$$V_j(z_j) \geq U_j \quad \text{for all } j \in \mathcal{J}.$$

In words, we are looking for the allocation that solves the Rawlsian social welfare problem among feasible deviating allocations which give the firm the maximum level of profits subject to the restriction that all types receive their reservation utility. Clearly a solution exists and one of the allocations which yields this is a pareto optimum.

In the optimum to this problem, if the reservation utility constraint is binding for a customer, then that customer receives no more than his baseline level of excess insurance. (Otherwise, reducing the customer's insurance to the baseline policy increases profits without violating any constraint). This means that in an optimum, if the reservation utility constraint is binding for a customer, then that customer cannot be pooled with a lower type, since the policy for the higher type would give the lower type less than his reservation utility.

Furthermore, if the reservation utility constraint is not binding for players of type worse than j, then those players must all be pooled with player j. To see this, start with the customer receiving the most excessive insurance; call him k. The firm's profits could be increased by lowering his policy unless either his reservation utility constraint is binding or the following self selection constraint is binding:

$$V_k(z_k) = V_k(z_j)$$

where j is the next higher type in $\mathcal{J}$. But in this case it is still cheaper to pool k at j's policy. Now suppose all customers of type lower than j are pooled with type j and the reservation utility constraint is not binding for any of them. Let ℓ be the next type in the set $\mathcal{J}$. Then either the reservation utility constraint is binding for type j or type j will be pooled with type ℓ. (If it were not binding, then the pooled policy could be reduced unless the next higher self selection constraint were binding -- and in the latter case, profits would increase if j were given ℓ's policy.)

We conclude that the only way for the reservation utility constraint for the lowest type in $\mathcal{J}$ not to be binding is for all customers to be pooled and therefore for the constraint to be binding for no customer. But then, by the fact that $\mathcal{S}_{-T}$ is comprehensive below, profits could be increased by reducing the pooled contract -- a contradiction.

It follows that the worst type in the deviation receives his reservation utility from a policy which provides no more insurance than the baseline policy. Since the baseline policy provided positive profits, this means that the worst type in the deviation is not subsidized. It is necessary by the previous lemma that all customers receive full or excessive insurance. By monotonicity of profits in the region of excessive insurance this means that no customer in $\mathcal{J}$ is subsidized.

The deviation that undermines (C,s) consists of the deviating firm adding the pareto optimal set to his menu of policies, and dropping y, and all deviating customers choosing their most preferred policy in the set. The constraints on the problem ensure that this is invulnerable to infiltration. The firm and all participants are better off than at (C,s). The final step is to show that this is a self-enforcing deviation.

Suppose not. Then there is a self enforcing deviation (Q,q) satisfying the no infiltration constraint with respect to customers not in Q. Define a new allocation for types in $\mathcal{J}$ where those not in Q are given the choice of any policy in either the deviating contract or the initial contract. By construction this allocation is IIC within $\mathcal{J}$ (The fact that it is IIC for the firm implies that no separated policy makes a loss; it is here that condition 2 on the Pareto optimum is vital.) Moreover the firm's profit is at least as large as in the deviation, because the customers not in Q did not wish to infiltrate. Thus we have found a feasible deviating allocation which weakly Pareto dominates the optimal Rawlsian allocation, and generates higher profit than π^*, a contradiction. This completes the proof for case II and the theorem.

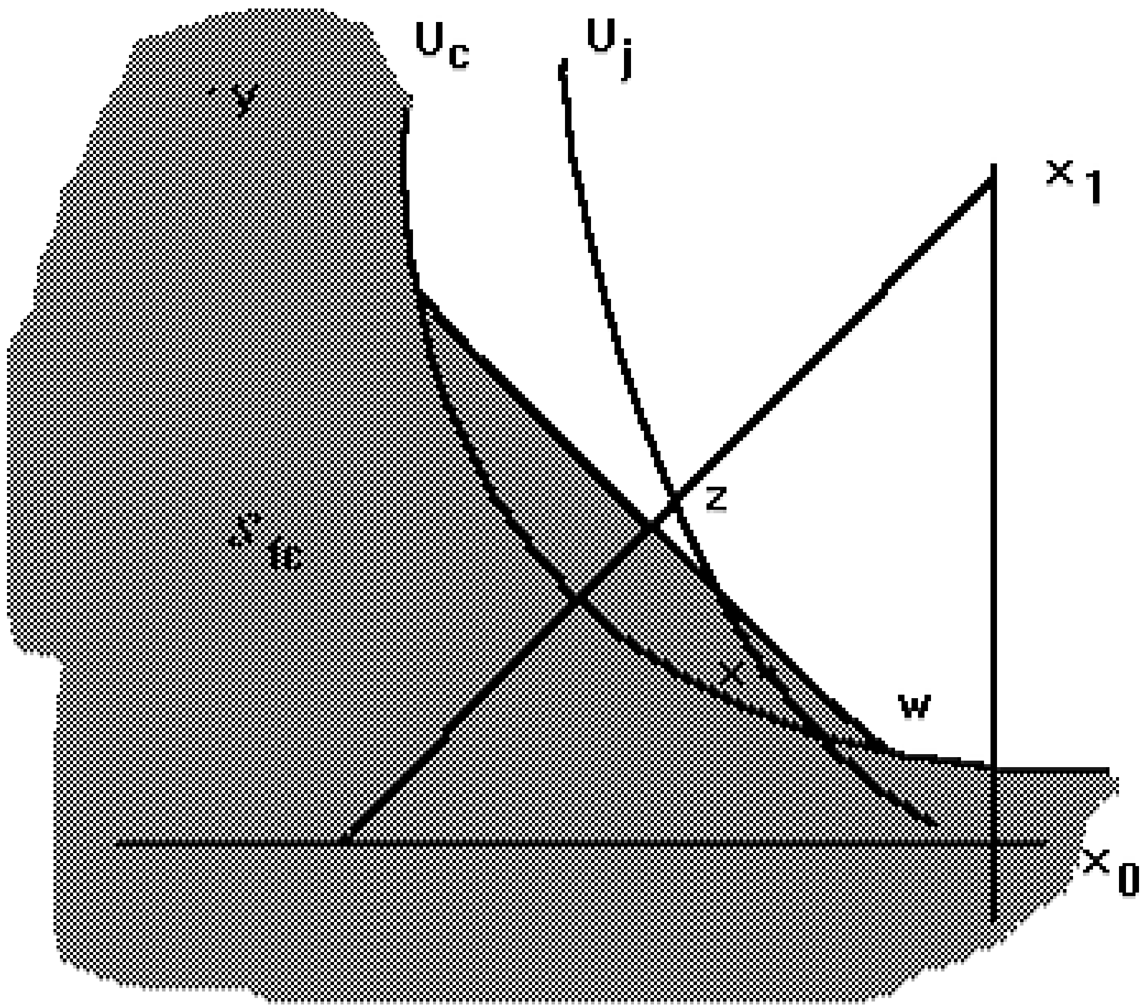
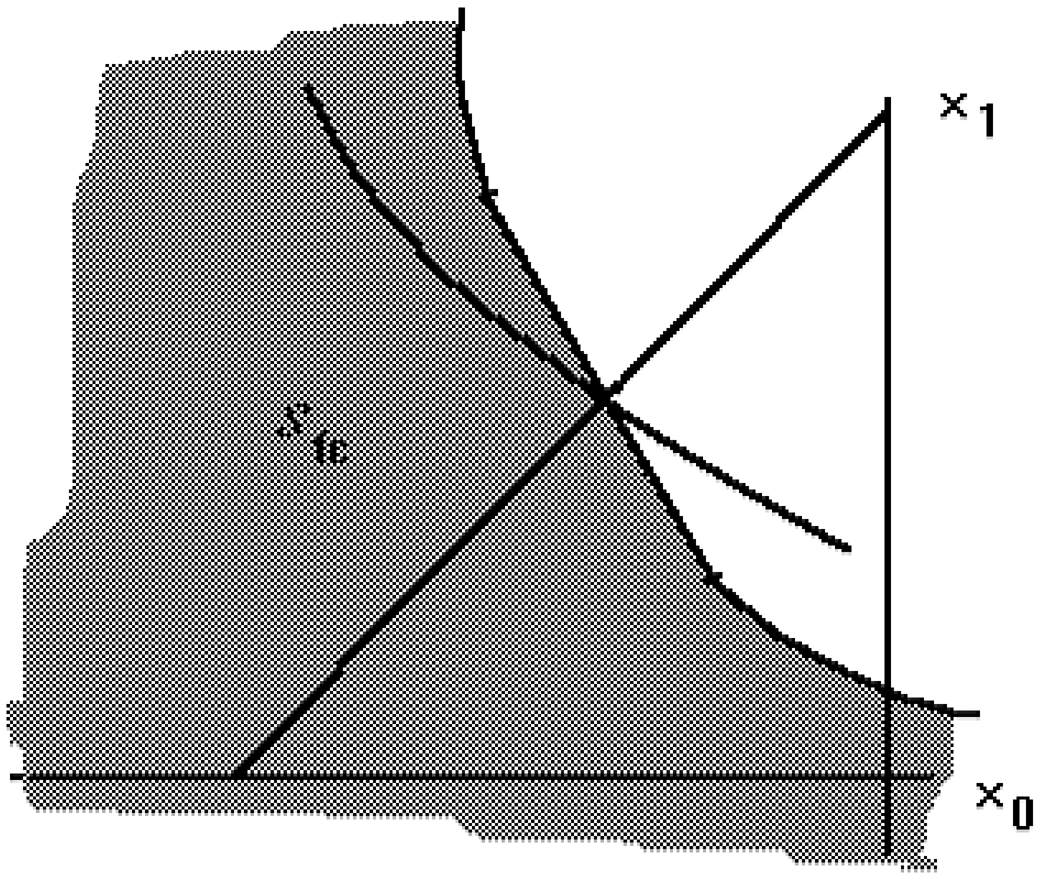


FIGURE A3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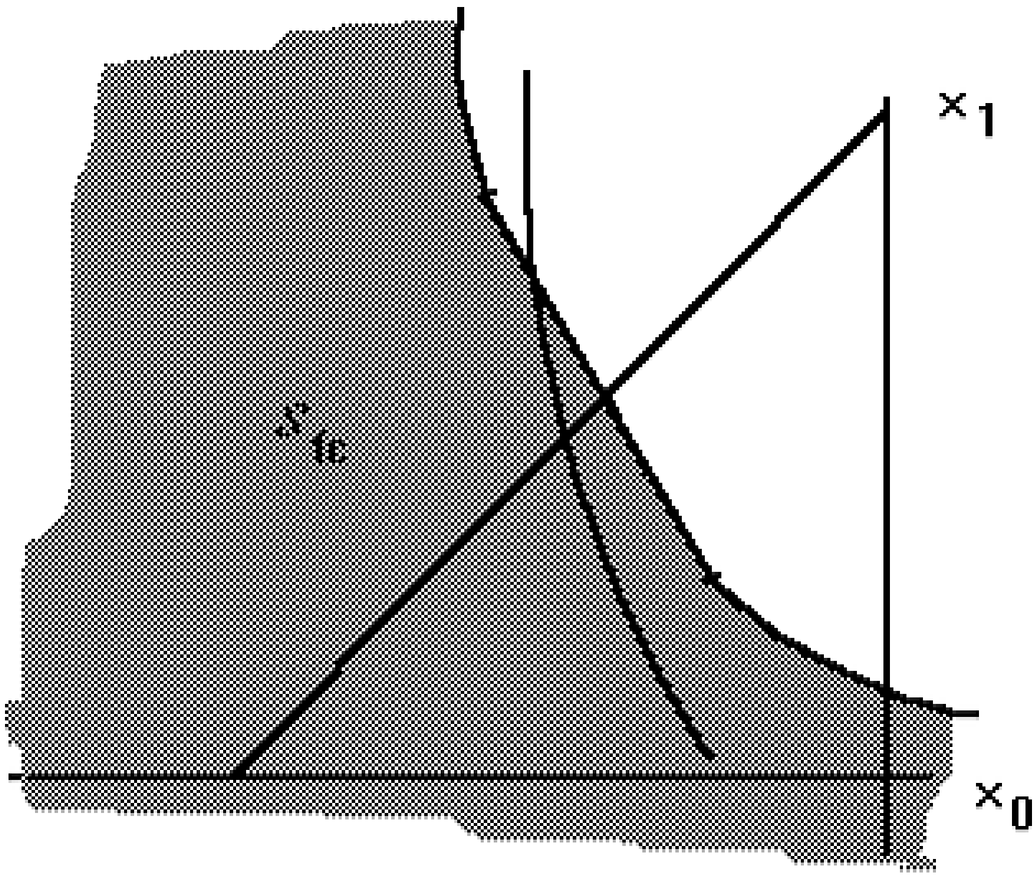

FIGURE A5