9. Appendix: Formal Results

Throughout the appendix we will use the notation $(x)_+$ as shorthand for $\max\{x, 0\}$. The two bidders are denoted by $i = \{1,\ 2\}$ and the two goods are denoted by $j = \{1, 2\}$. Bidder 1's valuation of object j will be denoted v_j. Bidder 2's valuation of object j will be denoted w_j. We will calculate expected profits from the point of view of bidder 1; therefore we let $\tilde{F}(w_1, w_2)$ denote bidder 1's subjective probability distribution over the valuations of bidder 2. We assume throughout that $\tilde{F}$ is a continuous distribution with support $[\underline{v}, \bar{v}]^2$. Further conditions on $\tilde{F}$ will be imposed as part of the various proofs in subsequent sections.

In determining whether a SPaR strategy vector is a Nash equilibrium, a central role is played by following function, the "expected gain to cooperative bidding":

$$\Gamma(v_1, v_2) = \int_{(w_1,w_2)\in S} (v_1 - R) - (v_1 - w_1)_+ - (v_2 - w_2)_+ \, d\tilde{F}(w_1, w_2). \tag{3}$$

To interpret this function, assume that when the outcome is coordinated, bidder 1 receives object 1 at the reserve price. Also assume that bidder 2 is following a SPaR strategy with a coordination region $S \subseteq [\underline{v}, \bar{v}]^2$. Then $\Gamma(v_1, v_2)$ is the expected increased payoff to bidder 1 in switching from competitive to cooperative bidding, given his valuations for the two goods.

The difference between the various kinds of equilibria described in the text is a difference in the region of coordination; for instance when $v_1 > v_2$ the coordination region for bidder 2 under a partially coordinated strategy with critical value C is $\{(w_1, w_2) \in [\underline{v}, \bar{v}]^2 \ | w_1 < \min\{C, w_2\}\}$. We begin with a lemma stating basic facts about the function Γ:

Lemma 10. *Suppose that S is a non-empty open set.*

a) The function Γ is concave in (v_1, v_2). The function is strictly increasing in v_1 for v_1 less than the supremum of w_1 in S, and constant in v_1 for v_1 greater than the supremum of w_1 in S. It is *strictly decreasing in* v_2 *for* v_2 *greater than the infimum of* w_2 *in* S.

b) If in addition

$$R \leq \inf_{(w_1,w_2)\in S} \min\{w_1, w_2\},$$

then (i) $\Gamma(R,R) = 0$ *and (ii)* $\Gamma(R+\varepsilon, R+\varepsilon) > 0$ *for* ε *positive and sufficiently small.*

Proof. Part a) follows from the fact that the integrand in (3) is concave, increasing in v_1 for $v_1 < w_1$, constant in v_1 for $v_1 > w_1$, decreasing in v_2 for $v_2 > w_2$ and constant in v_2 for $v_2 < w_2$. Part b) (i) is immediate. For b) (ii) we have:

$$\begin{aligned}\Gamma(R+\varepsilon, R+\varepsilon) &= \varepsilon \int_{(w_1,w_2)\in S} d\tilde{F}(w_1,w_2) - \int_{\substack{(w_1,w_2)\in S \\ w_1 < R+\varepsilon}} (R+\varepsilon - w_1)\, d\tilde{F}(w_1,w_2) \\ &\quad - \int_{\substack{(w_1,w_2)\in S \\ w_2 < R+\varepsilon}} (R+\varepsilon - w_2)\, d\tilde{F}(w_1,w_2). \\ &\geq \varepsilon \int_{(w_1,w_2)\in S} d\tilde{F}(w_1,w_2) - \varepsilon \int_{\substack{(w_1,w_2)\in S \\ w_1 < R+\varepsilon}} d\tilde{F}(w_1,w_2) \\ &\quad -\varepsilon \int_{\substack{(w_1,w_2)\in S \\ w_2 < R+\varepsilon}} d\tilde{F}(w_1,w_2)\end{aligned}$$

$$= \varepsilon(\Pr\{(w_1,w_2) \in S \cap w_1 > R+\varepsilon\} - \Pr\{(w_1,w_2) \in S \cap w_2 < R+\varepsilon\})$$

which is positive for ε positive and sufficiently small. ■

Until section 9.4, we assume that $R \leq \underline{v}$—that is, bidder 2 never places a valuation on either object lower than the reserve price.

9.1. Proof of Theorem 3. For this theorem we assume that $\tilde{F}(w_1,w_2) = F(w_1)F(w_2)$. That is, the valuations of each of the two objects are drawn independently from a continuous distribution F on $[\underline{v}, \bar{v}]$.

Suppose, without loss of generality, that $v_1 > v_2$. Then given his valuations, player 1 has four strategies in the first period: to bid on both objects, to bid on the higher valued object only, to bid on the lower valued object only, or to bid on no objects. The last is obviously inferior, the third is inferior to the second by an appeal to symmetry. Given that the other player is following the partially coordinated strategy, bidding on both objects guarantees that the outcome will be competitive. Bidding on only one object will yield that object at the reserve price if coordination is achieved, otherwise it will lead to competition. Thus coordination will be achieved if $w_2 > w_1$ and $C > w_1$, where $C = \bar{v}$ in the case of complete coordination and $\underline{v} < C < \bar{v}$ in the case of partial coordination.

In other words, the gain from bidding on on the higher valued object over bidding on both objects is

$$\Gamma(v_1, v_2; C) \equiv \iint\limits_{\substack{w_2 > w_1 \\ C > w_1}} [v_1 - R - (v_1 - w_1)_+ - (v_2 - w_2)_+] \, dF(w_1) \, dF(w_2). \tag{4}$$

(We will add the third argument to the function Γ when we wish to emphasize dependence on the size of the coordination region.) If we can show that for all (v_1, v_2) such that $\bar{v} \geq v_1 > v_2 \geq \underline{v}$, this expression is non negative when $C > v_2$, and non positive when $v_2 > C$, we will have proved the theorem. Lemma 10, part a) shows that Γ has the proper signs in the proper regions if and only if either

$$\Gamma(v, v; C) \geq 0 \text{ for all } v \text{ in the interval } (\underline{v}, C) \text{ and } \Gamma(C, C; C) = 0$$

or

$$\Gamma(v, v; \bar{v}) \geq 0 \text{ for all } v \text{ in the interval } (\underline{v}, \bar{v}) \text{ and } C = \bar{v}.$$

Since $\Gamma(v, v; C)$ is concave in v, lemma 10, part b) shows that $\Gamma(v, v; C) \geq 0$ for all v in the interval $(\underline{v}, C)$ if and only if $\Gamma(C, C; C) \geq 0$. Calcuation shows that

$$\Gamma(C, C; C) = -\int_{\underline{v}}^{C} F(v) \, dv + (C - R)F(C)\left(1 - \frac{1}{2}F(C)\right)$$

We conclude that there exists a partially coordinated strategy with critical value C provided

$$\Gamma(C, C; C) \geq 0 \text{ and } C \leq \bar{v}, \text{with at least one inequality strict.} \tag{5}$$

Note that $\Gamma(\underline{v}, \underline{v}; \underline{v})$ is equal to 0. If, per hypothesis, either $F'(\underline{v}) > 0$ and $F''(\underline{v})$ is finite or $R < \underline{v}$, then $\Gamma(\underline{v}, \underline{v}; \underline{v})$ is increasing (this is the only place where these hypothesized conditions are used in the proof). Thus either $\Gamma(C, C; C) \geq 0$ for all values of C in $(\underline{v}, \bar{v})$, in which case there is a fully coordinated equilibrium, or there is an interior value of C which supports a partially coordinated equilibrium. ■

Note that $\Gamma(C, C)$ depends only on the particulars of the distribution below C. Thus once we find a partially coordinated equilibrium we can make arbitrary changes to the distribution above C and we will continue to have a partly coordinated equilibrium. Also note that C need not be unique: while $\Gamma(v, v; C)$ is concave in v, $\Gamma(v, v; v)$ is not.

9.2. Proof of Theorem 2.

Corollary 11. *If $R \leq \underline{v}$, then there is a fully coordinated Nash equilibrium if and only if the distribution has an upper bound $\bar{v}$ such that the mean valuation is greater than or equal to $\frac{1}{2}(\bar{v} + R)$*

Proof. When $C = \bar{v}$, condition (5) reduces to:

$$\int_{\underline{v}}^{\bar{v}} v \, dF(v) \geq \frac{1}{2}(\bar{v} + R). \tag{6}$$

■

Theorem 12. *Suppose the support of $F(v)$ is $[\underline{v}, \bar{v}]$, $R \leq \underline{v}$, the mean valuation is greater than $(\bar{v} + R)/2$, and*

$$\limsup_{v \to R+} \frac{F(v)}{v - R} < \infty. \tag{7}$$

Then there exists a critical step size $\bar{\varepsilon} > 0$ such that for all step sizes smaller than $\bar{\varepsilon}$, the discrete version of the game has a fully coordinated Nash equilibrium.

Note that (7) holds automatically if $R < \underline{v}$.

Proof. As ε approaches zero, the payoffs in the discrete version of the game approach the payoffs of the continuous approximation. We will find a critical size such that for all pairs (v_1, v_2) in $[\underline{v}, \bar{v}]^2$, the errors in the estimated payoff are so small that the sign of the difference between the expected payoff from the competitive defection and the payoff from the coordinated strategy is unchanged by correcting for the error made by the continuous approximation. The proof would be immediate were $\Gamma(.,.)$ uniformly bounded away from zero on $[\underline{v}, \bar{v}]^2$. However, when $R = \underline{v}$, the costs of the defection approach zero as the v's become small. Thus the comparison becomes more delicate.

Without loss of generality, assume $v_1 \geq v_2$. Then from lemma 10, we know that $\Gamma(v_1, v_2) \geq \Gamma(v_1, v_1) \geq (v_1 - R)\Gamma(\bar{v}, \bar{v})/(\bar{v} - R) > 0$. The next to last inequality follows from the concavity of Γ, and the last inequality follows from the fact that by the hypothesis, condition (6) holds as a strict inequality.

In any realization of (v_1, v_2) and opponent's values (w_1, w_2) the realized payoff in the auction, given a competitive deviation, differs from the continuous valuation by no more than $\min\{\varepsilon, v_1 - R\}$ $+ \min\{\varepsilon, v_2 - R\}$. This claim is justified by the following considerations: The price that must be paid to obtain an object differs by at most ε. Thus if the object is obtained in both the approximation and the discrete game the difference in payoff is precisely the difference in price. Since the decision in both games as to whether to obtain the object is voluntary (if too expensive, the bidder can always say "no"), the only time that the decision will differ between the two versions of the game is when the player's valuation differs from the opponent's valuation by less than ε. Furthermore, when valuation v_j is less than $\varepsilon + R$, the payoff for object j will either be $v_j - R$ or 0 in the discrete version, and somewhere in the interval $[0, v_j - R]$ in the continuous approximation. Finally, if an object is not obtained in either version of the game, the error from the continuous approximation is zero.

Thus given (v_1, v_2), the absolute value of the error in the calculation of the expected payoff in the competitive deviation is no more than $F(v_1+\varepsilon)\min\{\varepsilon, v_1 - R\} + F(v_2+\varepsilon)\min\{\varepsilon, v_2 - R\}$. Finally, if the coordination succeeds, the error from the continuous approximation to that payoff is zero: the payoff is v_1 in either version of the game. Since there is a .5 probability of coordination, the absolute value of the error in the calculation of $\Gamma(v_1, v_2)$ is no more than half the error in the competitive deviation. Since $v_1 > v_2$, the expected error in calculating Γ is bounded by $F(v_1+\varepsilon)\min\{\varepsilon, v_1 - R\}$. Thus we need to find a critical value of $\bar{\varepsilon}$ such that for all smaller, positive ε, and for all $v_1 > R$,

$$F(v_1+\varepsilon)\min\{\varepsilon, v_1 - R\} < (v_1 - R)\frac{\Gamma(\bar{v}, \bar{v})}{\bar{v} - R}.$$

Define

$$h = \sup_{v \in (R, \bar{v}]} \frac{F(v)}{v - R}\ ;$$

h is finite by the continuity of F and by (7). Then the requirement is satisfied by

$$\bar{\varepsilon} = \frac{\Gamma(\bar{v}, \bar{v})}{2(\bar{v} - R)h}.$$

For if $v_1 - R \geq \varepsilon$, then

$$\frac{F(v_1+\varepsilon)\min\{\varepsilon, v_1 - R\}}{v_1 - R} = \frac{F(v_1+\varepsilon)\varepsilon}{v_1 - R} = \left(\frac{v_1 + \varepsilon - R}{v_1 - R}\right)\frac{F(v_1+\varepsilon)\varepsilon}{v_1 + \varepsilon - R} \leq 2h\varepsilon$$

and if $v_1 - R < \varepsilon$, then

$$\frac{F(v_1+\varepsilon)\min\{\varepsilon, v_1 - R\}}{v_1 - R} = F(v_1+\varepsilon) \leq F(R + 2\varepsilon) \leq 2h\varepsilon$$

and $2h\varepsilon < 2h\bar{\varepsilon} = \Gamma(\bar{v},\bar{v})/(\bar{v}-R)$. ■

The proof of Theorem 2 in the text then makes the extension from Nash equilibrium to subgame perfect Nash equilibrium.

Generalization to Dependent Distributions (Proof of Theorem 4). When the distribution of a player's two valuations is the symmetric distribution $\tilde{F}(w_1,w_2)$ the expected gain to competitive bidding is

$$\Gamma(v_1,v_2;C) = \iint_{\substack{w_2>w_1\\ C>w_1}} (v_1 - (v_1-w_1)_+ - (v_2-w_2)_+)\, d\tilde{F}(w_1,w_2)$$

As before, a coordinated equilibrium exists if and only if $\Gamma(\bar{v},\bar{v};\bar{v}) \geq 0$. In this case

$$\Gamma(\bar{v},\bar{v};\bar{v}) = \iint_{\substack{w_2>w_1\\ \bar{v}>w_1}} w_1\, d\tilde{F}(w_1,w_2) + \iint_{\substack{w_2>w_1\\ \bar{v}>w_2}} -\bar{v} + w_2\, d\tilde{F}(w_1,w_2)$$

$$\begin{aligned} &= -\bar{v}\Pr\{w_2>w_1\} + \iint_{w_2>w_1} w_1\, d\tilde{F}(w_1,w_2) + \iint_{w_2>w_1} w_2\, d\tilde{F}(w_1,w_2) \\ &= -\frac{\bar{v}}{2} + \iint w_1\, d\tilde{F}(w_1,w_2) \end{aligned}$$

(the last line exploits the symmetry between w_1 and w_2). In other words $\Gamma(\bar{v},\bar{v};\bar{v})$ is positive as long as the expectation of w_1 is greater than one half of $\bar{v}$. ■

9.3. Proof of Theorem 5. Let $\mathbf{H}$ be the compact space of *concave, non-decreasing* functions from $[R,\bar{v}]$ to itself.

Lemma 13. *Suppose that player 2 is following a partially coordinated strategy. Then there exists a function h in $\mathbf{H}$ such that player 1 has a best response which is a partially coordinated strategy where the coordination region is of the form*

$$\{(v_1,v_2) \in [\underline{v},\bar{v}]^2\, | v_2 \leq h(v_1)\}$$

Proof. By lemma 10, player 1's coordination region is a convex set in $[\underline{v}, \bar{v}]^2$ whose boundary is a non-decreasing, concave function. ■

Obviously the lemma holds for player 2 as well, with the subscripts on the goods swapped. The shaded areas in Figures 3a and b are typical coordination regions for players 1 and 2 respectively, with upper boundaries h_1 and h_2. We will identify each h in $\mathbf{H}$ with the corresponding partially coordinated strategy. We would therefore like to apply the Brouwer fixed point theorem to the set of strategy profiles in $\mathbf{H} \times \mathbf{H}$. Unfortunately the competitive strategy profile is a member of this set, since the competitive strategy corresponds to the function $h \equiv \underline{v}$. Thus we must find a closed subset of the form $\mathbf{H}_1 \times \mathbf{H}_2$, not containing competitive strategies and such that a best response to strategies in $\mathbf{H}_1$ is in $\mathbf{H}_2$ and *vice versa.* We will do this differently depending on whether R is less than or equal to $\underline{v}$.

We begin with the case where $R = \underline{v}$. We define a family of subsets $\mathbf{K}(C)$ as follows:

$$\mathbf{K}(C) = \{h \in \mathbf{H} | h(\bar{v}) \geq C\}$$

Our goal is to find C_1 and C_2, both greater than $\underline{v}$, such that when player 2's strategies is in $\mathbf{K}(C_2)$ player 1's best response is in $\mathbf{K}(C_1)$ and *vice versa.*

Lemma 14. *Suppose that $R = \underline{v}$ and C_1 and C_2 are both greater than $\underline{v}$. Define*

$$C_0 = \frac{C_1(C_2 - \underline{v}) + \underline{v}(\bar{v} - \underline{v})}{(C_2 - \underline{v}) + (\bar{v} - \underline{v})}. \tag{8}$$

If for all x in the interval $[C_0, C_1]$ the following condition holds

$$(x - \underline{v})\tilde{F}(x, \bar{v}) - \int_{\underline{v}}^{x} \tilde{F}(w_1, \bar{v})\, dw_1 - \int_{\underline{v}}^{C_1} \tilde{F}(x, w_2)\, dw_2 > 0 \tag{9}$$

then for any of player 2's strategies in $\mathbf{K}(C_2)$, player 1's best response is in $\mathbf{K}(C_1)$.

Proof. Let the function h_2 in $\mathbf{K}(C_2)$ denote player 2's strategy. Showing that the best response is in $\mathbf{K}(C_1)$ is equivalent to showing that $(\bar{v}, C_1)$ is in player 1's coordinating region, that is to say, that $\Gamma(\bar{v}, C_1)$ is non-negative. Now

$$\Gamma(\bar{v}, C_1) = \int_{w_2=\underline{v}}^{\bar{v}} \int_{w_1=\underline{v}}^{h_2(w_2)} w_1 - R - (C_1 - w_2)_+ \, d\tilde{F}(w_1, w_2). \tag{10}$$

Denote the expression on the left side of formula (9) by $Z(x)$. It can be rewritten as

$$Z(x) = \int_{w_2=\underline{v}}^{\bar{v}} \int_{w_1=\underline{v}}^{x} w_1 - R - (C_1 - w_2)_+ \, d\tilde{F}(w_1, w_2)$$

Thus we have

$$\Gamma(\bar{v}, C_1) - Z(x) = \int_{w_2=\underline{v}}^{\bar{v}} \int_{w_1=x}^{h_2(w_2)} w_1 - R - (C_1 - w_2)_+ \, d\tilde{F}(w_1, w_2) \tag{11}$$

Note that the common integrand is negative if and only if $w_1 + w_2 < R + C_1$. Suppose some x in the interval $[\underline{v}, \bar{v}]$ satisfies the following condition:

$$h_2(R + C_1 - x) = x \tag{12}$$

If $w_2 > R + C_1 - x$, then $h_2(w_2) > x$ and the integrand is positive for all w_1 in the interval $[x, h_2(w_2)]$. On the other hand if $w_2 < R + C_1 - x$, then $h_2(w_2) < x$ and the integrand is negative for all w_1 in the interval $[h_2(w_2), x]$. In other words, if x satisfies condition (12), then the expression in (11) is positive and $\Gamma(\bar{v}, C_1) > Z(x)$. Geometrically, condition (12) states that x is the value of h_2 at its intersection with the locus $w_1 + w_2 = R + C_1$. Moreover, if x satisfies (12) then x lies in the range $[C_0, C_1]$ (because the graph of h_2 lies between the line $w_1 = R$ and the line segment from (R, R) to the point $(\bar{v}, C_2)$, and the values C_1, C_0 respectively are the w_1 values of intersections with the locus $w_1 + w_2 = R + C_1$; see figure 4). Thus if there is an intersection and if $\Gamma(\bar{v}, C_1)$ is negative, then

$Z(x)$ must be negative at the intersection value of x, violating the assumption of the lemma. On the other hand if there is no x in the interval $[R, \bar{v}]$ which satisfies condition (12) then it must be that

$$h_2(\bar{v}) < R + C_1 - \bar{v}.$$

In this case, set x equal to $R + C_1 - \bar{v}$. Again it can be shown that x lies in the range $[C_0, C_1]$ and since $w_1 + w_2 < R + C_1$ for all (w_1, w_2) in the open rectangle $(\underline{v}, \bar{v}) \times (\underline{v}, x)$, the integrand is negative throughout this range and $Z(R+C_1-\bar{v})$ is negative, again violating the assumption of the lemma. ■

For any distribution it can be verified whether the conditions of the above lemma are satisfied for some C_1 and C_2. The rest of this section considers the case where the valuations of the two goods are independent, allowing us to describe simple sufficient conditions. Let player i's joint distribution over the two valuations be $F_{i1}(w_1)F_{i2}(w_2)$.

Lemma 15. *Let $R = \underline{v}$. Suppose both players' marginal distributions have densities f_{ij} which are twice differentiable in the neighborhood of the lower end of the support of the valuations. If $\underline{v} = 0$, suppose further that*

$$f_{11}(0)f_{22}(0) < \bar{v}^{-2}. \tag{13}$$

Then there exist $C_1, C_2 > 0$ such that for any of player 2's strategies in $\mathbf{K}(C_2)$, player 1's best response is in $\mathbf{K}(C_1)$ and vice versa.

Proof. In the case of independence, formula (9) reduces to

$$Z(x) = \int_{\underline{v}}^{C_0} (w_1 - R)\, dF_{21}(w_1) - F_{21}(x) \int_{\underline{v}}^{C_2} F_{22}(w_2)\, dw_2 > 0.$$

Since the expression is increasing in x whenever it is non negative, it is sufficient that it hold for $x = C_0$. Recalling that C_0 is also a function of C_1 and C_2, we search for a pair C_1, C_2 satisfying this inequality. Let $C_2 \to \underline{v}$ and let $C_1 = kC_2$. Then it suffices that

$$\lim_{C_2 \to \underline{v}} \frac{\int_{\underline{v}}^{C_0} w_1 \, dF_{21}(w_1)}{F_{21}(C_0) \int_{\underline{v}}^{C_0} F_{22}(w_2) \, dw_2} > 1$$

Both numerator and denominator, as well as their first derivatives, approach 0 as C_2 approaches $\underline{v}$. The second derivative of the denominator is also zero. If $\underline{v} > 0$, then let $k = 1$. The second derivative of the numerator is positive at $\underline{v}$ and we are done. On the other hand if $\underline{v} = 0$ the second derivative of the numerator is also zero. Evaluating the third derivatives of numerator and denominator at $C_2 = \underline{v} = 0$ yields the inequality

$$\frac{1}{2f_{22}(0)} \frac{d^2x}{dC_2}(0) = \frac{k}{\bar{v} f_{22}(0)} > 1$$

A similar analysis from the point of view of player 2 yields the inequality

$$\frac{1/k}{\bar{v} f_{11}(0)} > 1$$

There exists a k such that the two inequalities are simultaneously satisfied provided inequality (13) is satisfied. ■

In other words, when $R = \underline{v}$, minimal conditions imply that partial coordination by one bidder induces partial coordination by the other.

For the case $R < \underline{v}$, we jump immediately to the assumption of independent distributions. We define a family of subsets $\mathbf{J}(C) \subset \mathbf{H}$ as follows

$$\mathbf{J}(C) = \{h \in \mathbf{H} | h(\underline{v}) \geq C\}$$

Let C_j^* be defined by the following relationship:

$$\int_{\underline{v}}^{C_j^*} F_{jj}(w) \, dw = \underline{v} - R$$

Lemma 16. *Suppose $R < \underline{v}$. When bidder 2's uses a partially coordinated strategy, bidder 1's best response is in $\mathbf{J}(C_2^*)$, and vice versa.*

Proof. Let h_2 be bidder 2's strategy. It suffices to demonstrate that $\Gamma(\underline{v}, C_2^*) \geq 0$. Now

$$\Gamma\left(\underline{v}, C_2^*\right) = \int_{w_2=\underline{v}}^{\bar{v}} \int_{w_1=\underline{v}}^{h_2(w_2)} \underline{v} - R - (C_2^* - w_2)_+ \, dF_{22}(w_2) dF_{21}(w_1). \tag{14}$$

Define

$$Y\left(x\right) = \int_{w_2=\underline{v}}^{\bar{v}} \int_{w_1=\underline{v}}^{x} \underline{v} - R - (C_2^* - w_2)_+ \, dF_{22}(w_2) dF_{21}(w_1). \tag{15}$$

The common integrand in (14) and (15) is negative if and only if $w_2 < C_2^* + R - \underline{v}$. Thus if $h_2(C_2^* + R - \underline{v}) = x$, then $\Gamma\left(\underline{v}, C_2^*\right) > Y\left(x\right)$. But

$$Y\left(x\right) = F_{21}(x)[(\underline{v} - R) - \int_{w_2=\underline{v}}^{C_2^*} (C_2^* - w_2) \, dF_{22}(w_2)] = 0.$$

Therefore $\Gamma\left(\underline{v}, C_2^*\right)$ is positive. ■

Now we have subsets of $\mathbf{H} \times \mathbf{H}$ which exclude the competitive strategy. The final step is to verify continuity.

Lemma 17. *Suppose that the density of the distribution $\tilde{F}$ for player 2 exists and is bounded away from zero in a neighborhood of the point $(\underline{v}, \underline{v})$. Then player 1's best response to a strategy h of player 2 is a uniformly continuous function of $h(.)$ in $\mathbf{K}(C)$ or $\mathbf{J}(C), C > 0$.*

Proof. We now write $\Gamma(v_1, v_2; h)$ to emphasize the function's dependence on the opponent's strategy h. Given the function h let the function $g(.; h)$ be the best response of player 1. If $\Gamma(v_1, \underline{v}; h) < 0$, then $g\left(v_1; h\right) = \underline{v}$; if $\Gamma(v_1, \bar{v}; h) > 0$, then $g\left(v_1; h\right) = \bar{v}$. Otherwise, g is implicitly defined by $\Gamma(v_1, g\left(v_1; h\right); h) = 0$. Since the integrand in the formula (10) for Γ is uniformly bounded, Γ is uniformly continuous in h. We must show that for every $\varepsilon > 0$ there exists δ such that if $|\Gamma(v_1,v_2;a)-\Gamma(v_1,v_2;b)|\leq\delta$ for all $(v_1,v_2)\in[\underline{v},\bar{v}]^2$ then $|g(v_1;a)-g(v_1;b)|<\varepsilon$ for all v_1 in $[\underline{v},\bar{v}]$. This follows from the stronger claim: for every $\varepsilon>0$ there exists δ (independent of v_1,v_2,h) such that if $|\Gamma(v_1,v_2;h)|\leq\delta$, then there is a $\widehat{v}_2$ with $\Gamma(v_1,\widehat{v}_2;h)=0$ and $|\widehat{v}_2-v_2|<\varepsilon$. In fact, since Γ is decreasing in v_2, we will demonstrate the still stronger claim that for every $\varepsilon>0$ there exists $\delta>0$ such that if $\Gamma(v_1,v_2;h)>0$ and $v_2<\bar{v}-\varepsilon$, then $\Gamma(v_1,v_2+\varepsilon;h)\leq\Gamma(v_1,v_2;h)-\delta$, while if $\Gamma(v_1,v_2;h)<0$ and $v_2>\underline{v}+\varepsilon$, then $\Gamma(v_1,v_2-\varepsilon;h)\geq\Gamma(v_1,v_2;h)+\delta$. Since Γ is concave in v_2, we only need to demonstrate that for all $\varepsilon>0$ there exists $\delta>0$, such that $\Gamma(v_1,\underline{v}+\varepsilon;h)\leq\Gamma(v_1,\underline{v};h)-\delta$ uniformly in v_1,h. Let

$$\begin{aligned} t(w) &= R+\frac{C_2-R}{\bar{v}-R}(w-R) \qquad \text{if } \underline{v}=R \\ &= C_1^* \qquad \text{if } \underline{v}>R. \end{aligned}$$

(so that either way, $t(.)$ forms a lower bound on bidder 2's strategy $h(.)$). Now

$$\frac{\partial}{\partial v_2}\Gamma(v_1,v_2;h)=-\iint\limits_{\substack{w_1<h(w_2)\\ w_2<v_2}} d\tilde{F}(w_1,w_2)\leq-\iint\limits_{\substack{w_1<t(w_2)\\ w_2<v_2}} d\tilde{F}(w_1,w_2),$$

so that the partial derivative is negative for $v_2>\underline{v}$ and bounded away from zero, uniformly in v_1 and h. Call the bound $Q(v_2)$. Define

$$\delta=-\int_{\underline{v}}^{\underline{v}+\varepsilon} Q(v_2)\,dv_2$$

Then $\Gamma(v_1,\underline{v}+\varepsilon;h)-\Gamma(v_1,\underline{v};h)\leq-\delta$, and δ is independent of v_1,v_2, and h. ■

Given the above lemmas, theorem 5 follows by the Brouwer fixed point theorem.

9.4. Proof of Corollaries 7 and 8. Suppose the reserve price $R>\underline{v}$. If a bidder's valuation of each object is below R, he will bid on neither. If a bidder has one object with valuation above R, and one object with valuation below, he will bid on the higher valued object. Thus, without loss of generality, assume $v_1 > v_2 > R$. Suppose that when both of bidder 2's valuations are above R, he uses a partially coordinated strategy with critical value C. Then, in an auction with the no-excess-demand stopping rule, bidder 1's expected gain to cooperative bidding is

$$\Gamma(v_1, v_2; C) \equiv \iint_{\substack{\max\{w_2,R\}>w_1 \\ C>w_1}} (v_1 - R) - (v_1 - \max\{w_1, R\})_+ - (v_2 - \max\{w_2, R\})_+ \, dF(w_1)\, dF(w_2) \tag{16}$$

$$= \iint_{\substack{w_2>w_1 \\ C>w_1>R}} (v_1 - R) - (v_1 - w_1)_+ - (v_2 - w_2)_+ \, dF(w_1)\, dF(w_2) - F(R) \int_R^{v_2} F(w_2)\, dw_2 \tag{17}$$

(again the additional argument of the function Γ emphasizes its dependence on C).

Lemma 10 (for $S = \{(w_1, w_2) \in [v_1, v_2]^2 | w_2 > w_1, C > w_1 > R\}$), implies $\Gamma(R, R; C) = 0$, Γ is concave in (v_1, v_2), non decreasing in v_1, constant in v_1 for $v_1 > C$, and non increasing in v_2. Because of the final term in (17), we must expressly assume $\Gamma(R, R; C)$ is increasing in R; then, by the same logic as before, we can conclude that there is a partially coordinated equilibrium with critical value $C \in (R, \bar{v})$, if and only if

$$\Gamma(C, C; C) = 0, \text{ and } \Gamma(v, v; C) \text{ is increasing in } v \text{ at } v = R \tag{18}$$

and there is a fully coordinated equilibrium if and only if

$$\Gamma(\bar{v}, \bar{v}; \bar{v}) \geq 0, \text{ and } \Gamma(v, v; \bar{v}) \text{ is increasing in } v \text{ at } v = R. \tag{19}$$

Now direct calculation establishes that for v in the interval $[R, C]$

$$\Gamma(v, v; C) = \frac{1}{2} \int_R^v 2F(C) - F^2(C) - F^2(R) - 2F(r)\; dr.$$

Thus, a necessary condition for an equilibrium in partially coordinated strategies is

$$2F(C) - F^2(C) - F^2(R) - 2F(R) \geq 0 \tag{20}$$

which will be violated for all C if $F(R) \geq \sqrt{2} - 1$. This proves a slightly stronger version of the second claim of corollary 8.

To prove the first claim, suppose that, given R, there is a $C^* < \bar{v}$ such that

$$\Gamma(C^*, C^*; C^*) = 0$$

and

$$\Gamma(C, C; C) < 0 \text{ for all } C \in (C^*, \bar{v}].$$

Now

$$\begin{aligned} \frac{d\Gamma(C,C;C)}{dR} &= -\frac{1}{2}[2F(C) - F^2(C) - F^2(R) - 2F(R)] - \int_R^C F(r)\, dF(r) \\ &= -F(C) + F^2(R) - F(R) \end{aligned}$$

which by (20) is negative at $C = C^*$. Moreover, the expression is decreasing in C; thus $\Gamma(C, C; C)$ is a decreasing function of R for all $C \geq C^*$, meaning that C^* falls with increasing R.

In an auction with the no-price-increase stopping rule, bidder 1's expected gain to cooperative bidding is

$$\Gamma_P(v_1, v_2; C) = \Gamma(v_1, v_2; C) + (v_2 - R)F^2(R)$$

Suppose there is a partially coordinated equilibrium with critical value C^* under the no-excess-demand stopping rule. Then by (18) and (19), $\Gamma_P(C^*, C^*; C^*) > 0$. In the no-price-increase stopping rule auction a parallel analysis leads to the conditions for existence of coordinated and partially cooridinated equilibria which correspond to (18) and (19), substituting Γ_P for Γ. Since $\frac{d^2\Gamma_P(v,v,C)}{dv\, dC} > 0$, the no-price-increase conditions must be satisfied for some critical value greater than C^*, proving Corollary 7.

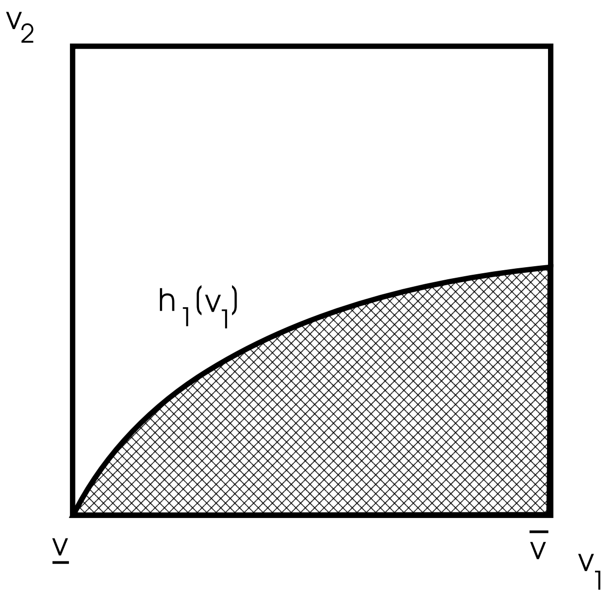

Figure 3a

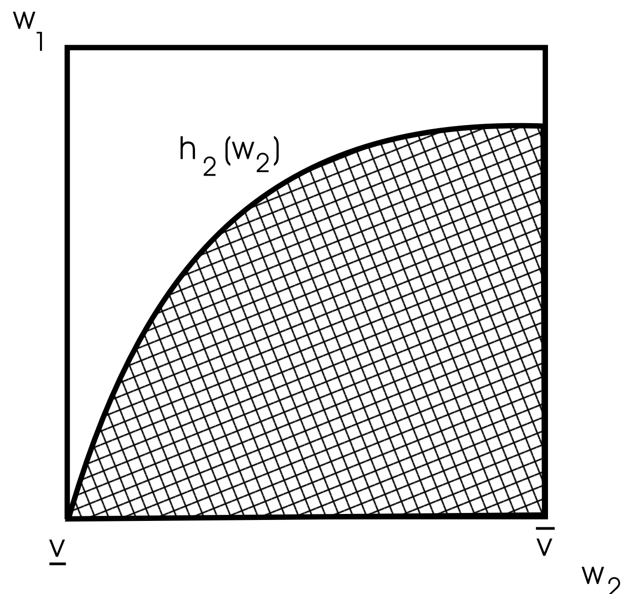

Figure 3b

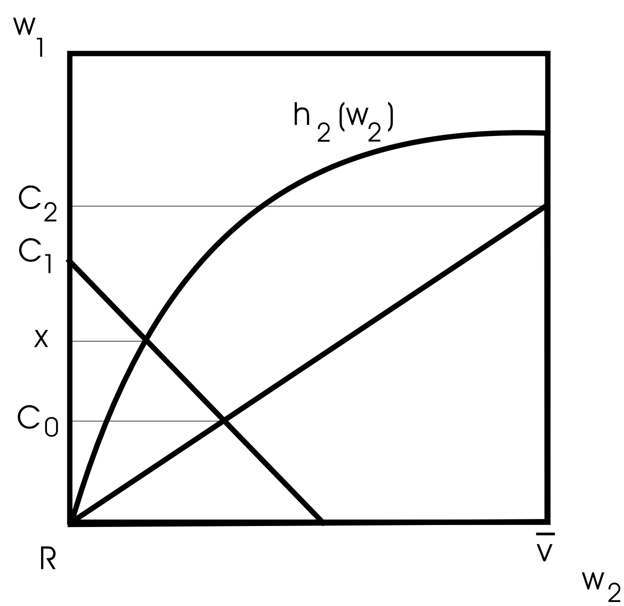

Figure 4